

THE PROJECTIONS OF n -KNOTS WHICH ARE NOT THE PROJECTION OF ANY UNKNOTTED KNOT

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Abstract. Let n be any integer greater than two. We prove that there exists a projection P having the following properties. (1) P is not the projection of any unknotted knot. (2) The singular point set of P consists of double points. (3) P is the projection of an n -knot which is diffeomorphic to the standard sphere.

We prove there exists an immersed n -sphere ($\subset \mathbb{R}^{n+1} \times \{0\}$) which is not the projection of any n -knot ($n > 2$). Note that the second theorem is different from the first one.

§1. INTRODUCTION AND MAIN RESULTS

The study of n -dimensional knots and links has a long history. The research was connected with surgery theory. (See [9], [20] etc. for the history.) There are many fruitful results. There are many outstanding problems. For example, n -links have not been classified. (This open problem is not discussed explicitly in this paper but it is one of motivations of this paper.)

When we study n -knots and n -links ($n \geq 2$), we sometimes see similarities and differences between the theory of 1-links and that of n -links. In this paper we point out one difference between them, associated with the projections of knots. (Theorem 1.3.).

When one studies classical knots (in \mathbb{R}^3), it is important to consider the projections of knots into \mathbb{R}^2 . See [1], [3], [8], [12], [16], [23], [25], etc. For 2-dimensional knots in \mathbb{R}^4 , one considers the projections of 2-knots into \mathbb{R}^3 . See [2], [4], [5], [7], [10], etc.

In order to state our problems (Problem 1.1 and 1.2) and our main theorem (Theorem 1.3.), we prepare some definitions on n -knots and on their projections.

We work in the smooth category.

An (*oriented*) n -(*dimensional*) knot K is a smooth oriented submanifold of $\mathbb{R}^{n+1} \times \mathbb{R}$ which is PL homeomorphic to the standard n -sphere. We say that n -knots K_1 and K_2 are *equivalent* if there exists an orientation preserving

Mathematics Subject Classification (1991): 57M25, 57Q45

This research was partially supported by Research Fellowships of the Promotion of Science for Young Scientists.

Keywords. The projections of n -knots, Kummer surface, spun knots

diffeomorphism $f : \mathbb{R}^{n+1} \times \mathbb{R} \rightarrow \mathbb{R}^{n+1} \times \mathbb{R}$ such that $f(K_1) = K_2$ and $f|_{K_1} : K_1 \rightarrow K_2$ is an orientation preserving diffeomorphism. We say that n -knots K_1 and K_2 are *identical* if $\text{id}(K_1) = K_2$ holds for the identity map $\text{id} : \mathbb{R}^{n+1} \times \mathbb{R} \rightarrow \mathbb{R}^{n+1} \times \mathbb{R}$ and $\text{id}|_{K_1} : K_1 \rightarrow K_2$ is an orientation preserving diffeomorphism.

Let T be the unit n -sphere in $\mathbb{R}^{n+1} \times \{0\} \subset \mathbb{R}^{n+1} \times \mathbb{R}$. An n -knot K is said to be *unknotted* if K is equivalent to T .

Let $\pi : \mathbb{R}^{n+1} \times \mathbb{R} \rightarrow \mathbb{R}^{n+1}$ be the natural projection map. We assume $\pi|_K$ is a self-transverse immersion. The *projection* P of an n -knot K is $\pi|_K(K)$ in \mathbb{R}^{n+1} . We give P an orientation by using the orientation of K . The *singular point set* of the projection of an n -knot K is the set $\{x \in \pi|_K(K) \mid \#\{(\pi|_K)^{-1}(x)\} \geq 2\}$.

It is well-known that the projection of any 1-dimensional knot is the projection of an unknotted 1-knot. This fact is used in the definition of the Jones polynomial. It is also used in some definitions of the Conway-Alexander polynomial. See, e.g., [13], and [14].

It is natural to ask the following question.

Problem 1.1. Let P be the projection of an n -knot K . Let K be diffeomorphic to the standard sphere. Is P necessarily the projection of an unknotted n -knot?

In this paper we give a negative answer to Problem 1.1 when $n > 2$.

The problem in the case $n = 2$ is still open. As mentioned above, the answer in the case $n = 1$ is affirmative.

Note that, when $n = 1$, the singular point set always consists of double points.

Here, we consider the following problem 1.2. Obviously, if the answer to Problem 1.2 is negative, the answer to Problem 1.1 is negative. Of course, as mentioned above, the answer to Problem 1.2 for $n = 1$ is affirmative.

Problem 1.2. Let P be the projection of an n -knot K . Let K be diffeomorphic to the standard sphere. Suppose that the singular point set of P consists of double points. Then does it follow that P is the projection of an unknotted n -knot?

In this paper we prove that the answer to Problem 1.2 in the case $n \geq 3$ is negative and hence the answer to Problem 1.1 in the case $n \geq 3$ is also negative.

Theorem 1.3. Let n be any integer greater than two. Then there exists a projection P having the following properties.

- (1) P is not the projection of any unknotted knot.
- (2) The singular point set of P consists of double points.
- (3) P is the projection of an n -knot which is diffeomorphic to the standard sphere.

In §4 we prove there exists an immersed n -sphere ($\subset \mathbb{R}^{n+1} \times \{0\}$) which is not the projection of any n -knot ($n > 2$) (Theorem 4.1). Note that this theorem is different from Theorem 1.3.

§2. PROOF OF THEOREM 1.3 IN THE CASE $n = 3$

Let V be the punctured Kummer surface. It is known that V has the following

properties. See, e.g., p.9 of [15].

- (1) The intersection matrix is $\oplus^3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus^2 E_8$. The signature $\sigma(V)$ is equal to -16 .
- (2) There exists a handle decomposition with one 0-handle and twenty-two 2-handles.

We prove:

Lemma 2.1. *There exist an immersion $\alpha : V \looparrowright \mathbb{R}^4$ and charts (U_i, ϕ_i) of \mathbb{R}^4 with the following properties. ($i = 1, \dots, \mu$, where μ is a positive integer.)*

- (i) $U_1 \cup \dots \cup U_\mu$ covers the singular point set of $\alpha(V)$.
- (ii) $U_i \cap U_j = \emptyset$ for $i \neq j$.
- (iii) $\phi_i : U_i \cong \{(x, y, z, w) | x^2 + y^2 + z^2 + w^2 < 1\}$.
- (iv) $\alpha^{-1}(U_i)$ consists of two components V_{1i} and V_{2i} and $\phi_i \circ \alpha|_{V_{1i}}$ and $\phi_i \circ \alpha|_{V_{2i}}$

are embeddings. Furthermore:

$$\phi_i \circ \alpha(V_{1i}) = \{(x, y, z, w) | x^2 + y^2 < 1, z^2 + w^2 \leq \frac{1}{4}\}. \quad \phi_i \circ \alpha(V_{2i}) = \{(x, y, z, w) | x^2 + y^2 \leq \frac{1}{4}, z^2 + w^2 < 1\}.$$

Proof. Take the handle decomposition of $V = h^0 \cup (\bigcup_{i=1}^{22} h_i^2)$ associated with the framed link given on p.9 of [15]. Let $L_V = (K_{V,1}, \dots, K_{V,22})$ denote this framed link. As described in [15], the framing of $K_{V,1}$ is zero and the framing of $K_{V,i}$ is -2 for $i \neq 1$. The underlying link is also denoted by L_V . The framing of attaching h_i^2 is that of $K_{V,1}$.

Consider \mathbb{R}^4 as $\mathbb{R}^3 \times \{t | t \in \mathbb{R}\}$. Take an embedded closed 4-ball B^4 in $\mathbb{R}^3 \times \{t | t \in \mathbb{R}\}$ so that $B^4 \subset \mathbb{R}^3 \times \{t | t \leq 0\}$ and $B^4 \cap (\mathbb{R}^3 \times \{t | t = 0\})$ is the 3-ball D^3 . Take L_V in $\text{Int } D^3$.

Recall the following. Suppose \mathbb{R}_1^2 and \mathbb{R}_2^2 intersect transversely in \mathbb{R}^4 . Suppose $\mathbb{R}_1^2, \mathbb{R}_2^2, \mathbb{R}^4$ are oriented. Suppose $\mathbb{R}_1^2 \cap \mathbb{R}_2^2$ is one point p . Then we can define the signature $\sigma(p)$ for p by using the orientations of $\mathbb{R}_1^2, \mathbb{R}_2^2, \mathbb{R}^4$.

Sublemma. *There exists a self-transverse immersion $f : D_1^2 \amalg \dots \amalg D_{22}^2 \looparrowright \mathbb{R}^3 \times [0, \infty)$ with the following properties.*

- (1) f is transverse to $\mathbb{R}^3 \times \{0\}$. $f(D_1^2 \amalg \dots \amalg D_{22}^2) \cap \mathbb{R}^3 \times \{0\} = f(\partial D_1^2 \amalg \dots \amalg \partial D_{22}^2)$.
- (2) $f|_{\partial D_1^2 \amalg \dots \amalg \partial D_{22}^2}$ is an embedding.
- (3) $f(\partial D_1^2 \amalg \dots \amalg \partial D_{22}^2)$ in $\mathbb{R}^3 \times \{0\}$ is L_V .
- (4) Let p_{ij} be the singular points of $f(D_i^2)$. We have

$$\Sigma_j \sigma(p_{ij}) = \begin{cases} 0 & \text{for } i = 1 \\ -1 & \text{for } i = 2, \dots, 22. \end{cases}$$

Proof. Since $\pi_1(\mathbb{R}^3 \times [0, 1]) = 1$, there is a map $g : D_1^2 \amalg \dots \amalg D_{22}^2 \looparrowright \mathbb{R}^3 \times [0, \infty)$ such that $g(\partial D_i^2) = K_{V,i}$. Perturb g and make g self-transverse. Let q_{ik} be the singular points of $g(D_i^2)$.

We prove: There is a self-transverse immersed 2-sphere $S_\varepsilon \subset \mathbb{R}^3 \times (0, \infty)$ ($\varepsilon = +1, -1$) such that the singular point set is one point and its signature is ε .

Because: Take $X = \{(a, b, c, d) | a^2 + b^2 + c^2 + d^2 \leq 1\} \subset \mathbb{R}^3 \times (0, \infty)$. Put $D_1^2 = \{(a, b, c, d) | a^2 + b^2 \leq 1, c = d = 0\}$. Put $D_2^2 = \{(a, b, c, d) | a = b = 0, c^2 + d^2 \leq 1\}$. Then $H = (\partial D_1^2, \partial D_2^2)$ in the 3-sphere ∂X is the Hopf link. Take a Seifert surface A for H which is diffeomorphic to $S^1 \times [-1, 1]$. Then $D_1 \cup A \cup D_2$ is an example of S_{+1}^2 . Put $D_1^2 = \{(a, b, c, d) | a^2 + c^2 \leq 1, b = d = 0\}$ and $D_2^2 = \{(a, b, c, d) | a = c = 0, b^2 + d^2 \leq 1\}$ instead. Then we obtain S_{-1}^2 . This completes the proof.

We continue the proof of Sublemma. Take a connected sum of S_ε^2 and $g(D_\mu^2)$. Then we obtain an immersed 2-disc D . Put $g' : D_1^2 \amalg \dots \amalg D_{22}^2 \looparrowright \mathbb{R}^3 \times [0, \infty)$ so that $g'(D_i^2) = g(D_i^2)$ for $i \neq \mu$ and $g'(D_\mu^2) = D$. Let $q'_{ik'}$ be the singular points of $g'(D_i^2)$. Then $\Sigma_{k'} \sigma(q'_{ik'}) = \Sigma_k \sigma(q_{ik})$ for $i \neq \mu$ and $\Sigma_{k'} \sigma(q'_{\mu k'}) = \Sigma_k \sigma(q_{\mu k}) + \varepsilon$.

We make f in Sublemma from the above g . Suppose $\Sigma_k \sigma(q_{ik}) \neq \Sigma_j \sigma(p_{ij})$. By using immersed spheres as S_ε^2 and using operations as above, we modify g to obtain f . This completes the proof of Sublemma.

Let ν_i be the normal D^2 -bundle of D_i^2 in $\mathbb{R}^3 \times [0, \infty)$. Let E_i be the total space of ν_i . Note $E_i \cong D_i^2 \times D^2$. Here, we have an immersion $\beta_i : E_i \looparrowright \mathbb{R}^3 \times \{0\}$. Note $\beta_i(E_i) \cap B^4 = (\partial D_i^2) \times D^2 \subset \mathbb{R}^3 \times \{0\}$. Let $W = B^4 \cup (\bigcup_{i=1}^{22} E_i)$. Here, we have an immersion $\beta : W = B^4 \cup (\bigcup_{i=1}^{22} E_i) \looparrowright \mathbb{R}^4$ such that $\beta(B^4) = B^4$ and $\beta(E_i) = \beta_i(E_i)$. Note we can regard $\bigcup_{i=1}^{22} E_i$ as 4-dimensional 2-handles attached to B^4 along L_V . E_i is called h_i . We write $W = h^0 \cup (\bigcup_{i=1}^{22} h_i^2)$.

We prove: The framing of attaching h_i^2 is $2 \cdot \Sigma_j \sigma(p_{ij})$.

Proof. We perturb $D_i^2 = D_i^2 \times \{0\}$ in $h_i^2 = E_i = D_i^2 \times D^2$ so that we perturb ∂D_i^2 in ∂h_i^2 . The result is called \bar{D}_i^2 . Suppose D_i^2 and \bar{D}_i^2 intersect transversely. Suppose $(\partial D_i^2, \partial \bar{D}_i^2)$ in $\mathbb{R}^3 \times \{0\}$ is the trivial link. By the definition of the framing, the algebraic intersection number of $D_i^2 \cap \bar{D}_i^2$ is the framing. Note that the points $D_i^2 \cap \bar{D}_i^2$ exist around the points p_{ij} . The algebraic intersection number is $2 \cdot \Sigma_j \sigma(p_{ij})$. This completes the proof.

Hence $W = h^0 \cup (\bigcup_{i=1}^{22} h_i^2)$ is V . This completes the proof of Lemma 2.1.

We use Lemma 2.1 and prove the following Proposition 2.2.

By the definition of 3-knots, all 3-knots are diffeomorphic to the standard sphere. Hence Proposition 2.2 induces Theorem 1.3 in the case $n = 3$.

Proposition 2.2. *Let r be any integer. Then there exists a projection P having the following properties.*

- (1) *If the projection of K is P , K is not unknotted.*
- (2) *If the projection of K is P , $\sigma(K) = 16r$.*
- (3) *The singular point set of P consists of double points.*

Note. An integer σ is the signature of a 3-knot if and only if σ is a multiple of sixteen. See, e.g., §10 of [19].

Proof of Proposition 2.2. We first prove Proposition 2.2 when $r = -1$.

Consider $\alpha : V \looparrowright \mathbb{R}^4$ in Lemma 2.1 as $\alpha : V \looparrowright \mathbb{R}^4 \times \{0\}$ ($\subset \mathbb{R}^4 \times \mathbb{R}$). Let β be the self-transverse immersion $\alpha|_{\partial V = S^3} : S^3 \looparrowright \mathbb{R}^4 \times \{0\}$. We prove

Lemma. $\beta(S^3)$ is the projection of a 3-knot ($\subset \mathbb{R}^4 \times \mathbb{R}$).

Proof. Let $\pi : \mathbb{R}^4 \times \mathbb{R} \rightarrow \mathbb{R}^4 \times \{0\}$ be the natural projection map. There is a submanifold B which is diffeomorphic to V such that $\pi(B) = \alpha(V)$. Because: The immersion α is an embedding in $\{\mathbb{R}^4 - (U_1 \cup \dots \cup U_\mu)\} \times \mathbb{R}$. In each U_i , push $\alpha(V_{1i})$ into the direction of $\mathbb{R}^4 \times \{t|t > 0\}$ (or $\mathbb{R}^4 \times \{t|t < 0\}$) so that $\alpha(V_{2i})$ is fixed. Thus we obtain a submanifold B which is diffeomorphic to V such that $\pi(B) = \alpha(V)$.

Hence $\beta(S^3)$ is the projection of the 3-knot ∂B . This completes the proof of the Lemma.

Let P denote the projection $\beta(S^3)$.

Claim 2.3. P satisfies conditions (1)(2)(3) of Proposition 2.2 when $r = -1$.

Proof of Claim 2.3. Let Q be the singular point set of P . By the construction of α and β , we have

$$(1) Q \subset (U_1 \cup \dots \cup U_\mu) \times \{t = 0\}$$

(2) $Q \cap U_i$ is $S^1 \times S^1$. (In particular, see Lemma 2.1(iv).) Hence $Q \cap U_i$ is connected.

$$(3) Q \cap U_i \text{ consists of double points.}$$

Hence P satisfies condition (3) of Proposition 2.2.

Let A be a 3-knot whose projection is P . Put $W_{1i} = (\pi|A)^{-1}(\partial V_{1i})$ and $W_{2i} = (\pi|A)^{-1}(\partial V_{2i})$. Recall that $P \supset \partial V_{1i}$ and $P \supset \partial V_{2i}$. Here, we write V_{ji} for $\alpha(V_{ji})$.

We assign to the 3-knot A an element $\rho(A) \in \mathbb{Z}_2^\mu$ given as follows. Let $\mathbb{Z}_2 = \{+1, -1\}$. If W_{1i} is over (resp. under) W_{2i} , then we define the i -th coordinate of $\rho(A)$ to be $+1$ (resp. -1). We define ‘over’ and ‘under’ by using the direction of \mathbb{R}_t .

Note the following. (1) Let A' be a 3-knot whose projection is P . If $\rho(A') = \rho(A)$, then A' is equivalent to A . (2) For any element $x \in \mathbb{Z}_2^\mu$, there is such a 3-knot K with $\rho(K) = x$.

Let B be a submanifold which is diffeomorphic to V such that $\pi(B) = \alpha(V)$. Put $X_{1i} = (\pi|A)^{-1}(V_{1i})$ and $X_{2i} = (\pi|A)^{-1}(V_{2i})$. We give the submanifold B an element $\rho(B) \in \mathbb{Z}_2^\mu$ as follows. If X_{1i} is over (resp. under) X_{2i} , then we define the i -th coordinate of $\rho(B)$ to be $+1$ (resp. -1).

Note the following. (1) Let B' be a submanifold which is diffeomorphic to V such that $\pi(B') = \alpha(V)$. If $\rho(B') = \rho(B)$, then the submanifold B' is equivalent to the submanifold B . (2) For any element $x \in \mathbb{Z}_2^\mu$, there is such a submanifold B with $\rho(B) = x$.

Take a 3-knot A whose projection is P . Then there is a submanifold B which is diffeomorphic to V such that $\rho(B) = \rho(A)$. Since $\rho(B) = \rho(\partial B)$, $\rho(\partial B) = \rho(A)$. Hence the 3-knot A is equivalent to the 3-knot ∂B . Therefore A has a Seifert hypersurface which is diffeomorphic to V . Hence the signature of A is -16 . Therefore A is knotted. Hence P satisfies conditions (1) (2) of Proposition 2.2.

This completes the proof of Claim 2.3 and thus the proof of Proposition 2.2 in the case $r = -1$. We next prove Proposition 2.2 when $r \neq -1$. We divided the proof into the three cases, $r < -1$, $r = 0$, $r \geq 1$.

Let $-P$ denote what we obtain from P when we give the opposite orientation to P . Let P^* denote what we obtain from P when we give the opposite orientation to S^3 .

We prove Proposition 2.2 when $r < -1$. Let \bar{P} be an immersed 3-sphere in $\mathbb{R}^4 \times \{0\} = \mathbb{R}^3 \times \mathbb{R}_w \times \{0\}$. Suppose $\bar{P} \subset \mathbb{R}^3 \times \{w|0 \leq w \leq |r|\} \times \{0\}$. For each $s \in \Lambda = \{0, 1, \dots, |r|\}$, suppose that $\bar{P} \cap (\mathbb{R}^3 \times \{w=s\} \times \{0\})$ is a 2-sphere which bounds a 3-disc D_s embedded in $\mathbb{R}^3 \times \{w=s\} \times \{0\}$. If $s, s+1 \in \Lambda$, put $P_{s+1} = \bar{P} \cap (\mathbb{R}^3 \times \{s \leq w \leq s+1\} \times \{0\})$. Suppose $P_{s+1} \cup D_s^3 \cup D_{s+1}^3$ is a parallel displacement of P . Then \bar{P} is the projection of a 3-knot. So \bar{P} satisfies condition (3) of Proposition 2.2. It also follows that the 3-knots with projection \bar{P} are of the form $K_1 \# \dots \# K_{|r|}$, where the projection of K_* is P . Note that $\sigma(K_1 \# \dots \# K_{|r|}) = \sum_{*=1}^{|r|} \sigma(K_*) = -16|r| = 16r$. Hence \bar{P} satisfies condition (2) of Proposition 2.2. Also each $K_1 \# \dots \# K_{|r|}$ is not unknotted. Hence \bar{P} satisfies condition (1) of Proposition 2.2. This completes the proof of Proposition 2.2 in the case of $r < -1$.

We now prove Proposition 2.2 when $r \geq 1$. Let \bar{P} be as above. Take $-\bar{P}$. Then $-\bar{P}$ satisfies the conditions of Proposition 2.2 when $r \geq 1$.

Finally, we prove Proposition 2.2 when $r = 0$. Let \hat{P} be an immersed 3-sphere in $\mathbb{R}^4 \times \{0\} = \mathbb{R}^3 \times \mathbb{R}_w \times \{0\}$. Suppose $\hat{P} \cap (\mathbb{R}^3 \times \{w=0\} \times \{0\})$ is a 2-sphere which bounds a 3-disc D embedded in $\mathbb{R}^3 \times \{w=0\} \times \{0\}$. Suppose $[\hat{P} \cap (\mathbb{R}^3 \times \{w \geq 0\} \times \{0\})] \cup D$ is a parallel displacement of P and $[\hat{P} \cap (\mathbb{R}^3 \times \{w \leq 0\} \times \{0\})] \cup D$ is a parallel displacement of $-P^*$. Then \hat{P} is the projection of a 3-knot. So \hat{P} satisfies condition (3) of Proposition 2.2. It also follows that all 3-knots with projection \hat{P} are of the form $K_1 \# (-K_2^*)$, where the projection of K_* is P . Note that $\sigma(K_1 \# (-K_2^*)) = \sigma(K_1) + \sigma(-K_2^*) = 0$. Hence \hat{P} satisfies condition (2) of Proposition 2.2.

In order to prove that \hat{P} satisfies condition (1) of Proposition 2.2, we prove that each $K_1 \# (-K_2^*)$ is not unknotted. We begin by recalling the following fact. See, e.g., §14 of [19] for the Alexander polynomials. See, e.g., §6 of [19] for simple knots. The author gives a proof in the appendix.

Theorem 2.4 (known). *Let K be a simple $(2k+1)$ -knot ($k \geq 1$). Then the Alexander polynomial of K is trivial if and only if K is unknotted.*

K_1 and K_2 are 3-knots whose projections are P . Then K_i bounds a Seifert hypersurface diffeomorphic to the punctured Kummer surface V . Hence K_i bounds a simply connected Seifert hypersurface. So K_i is a simple knot. Since $\sigma(V) \neq 0$, K_1 is not unknotted and $-K_2^*$ is not unknotted. Therefore the Alexander polynomial $\Delta_{K_1}(t)$ of K_2 is nontrivial and the Alexander polynomial $\Delta_{-K_2^*}(t)$ of $-K_2^*$ is also nontrivial.

Recall the following fact.

Theorem (known). *Let K_1 and K_2 be $(2k+1)$ -knots ($k \geq 1$). Suppose the Alexander polynomial of each K_i is nontrivial. Then the Alexander polynomial of $K_1 \# K_2$ is also nontrivial.*

From this we conclude that each $K_1 \# (-K_2^*)$ is not unknotted.

This proves condition (1) of Proposition 2.2 in the case $r = 0$ and thus completes the proof of Proposition 2.2. We conclude that Theorem 1.3 holds in the case of $n = 3$.

§3. PROOF OF THEOREM 1.3 IN THE CASE $n > 3$

We define the *spun projection* of a projection. Let P be the projection of an n -knot K . Suppose $K \subset \mathbb{R}^n \times \mathbb{R}_t \times \mathbb{R}_u$ and $P \subset \mathbb{R}^n \times \{t = 0\} \times \mathbb{R}_u$. Let $\pi : \mathbb{R}^n \times \mathbb{R}_t \times \mathbb{R}_u \rightarrow \mathbb{R}^n \times \{t = 0\} \times \mathbb{R}_u$ be the natural projection map. We suppose $P \subset \mathbb{R}^n \times \{t = 0\} \times \{u \geq 0\}$ and that $P \cap (\mathbb{R}^n \times \{t = 0\} \times \{u = 0\})$ is an n -disc D embedded in $\mathbb{R}^n \times \{t = 0\} \times \{u = 0\}$. Suppose D does not intersect the singular point set Q of P .

Take $(\mathbb{R}^n \times \mathbb{R}_t \times \mathbb{R}_u) \times \mathbb{R}_v$ and we regard $\mathbb{R}^n \times \mathbb{R}_t \times \mathbb{R}_u$ as $\mathbb{R}^n \times \mathbb{R}_t \times \mathbb{R}_u \times \{v = 0\}$. We regard $\mathbb{R}^n \times \mathbb{R}_t \times \mathbb{R}_u \times \mathbb{R}_v$ as the result of rotating $\mathbb{R}^n \times \mathbb{R}_t \times \{u \geq 0\} \times \{v = 0\}$ around the axis $\mathbb{R}^n \times \mathbb{R}_t \times \{u = 0\} \times \{v = 0\}$.

Then we regard $\mathbb{R}^n \times \{t = 0\} \times \mathbb{R}_u \times \mathbb{R}_v$ as the result of rotating $\mathbb{R}^n \times \{t = 0\} \times \{u \geq 0\} \times \{v = 0\}$ around the axis $\mathbb{R}^n \times \{t = 0\} \times \{u = 0\} \times \{v = 0\}$.

Let $\pi : \mathbb{R}^n \times \mathbb{R}_t \times \mathbb{R}_u \times \mathbb{R}_v \rightarrow \mathbb{R}^n \times \{t = 0\} \times \mathbb{R}_u \times \mathbb{R}_v$ be the natural projection map. We can regard P as a subset of $\mathbb{R}^n \times \{t = 0\} \times \{u \geq 0\} \times \{v = 0\}$ since $\mathbb{R}^n \times \{t = 0\} \times \{u \geq 0\}$ is identified with $\mathbb{R}^n \times \{t = 0\} \times \{u \geq 0\} \times \{v = 0\}$.

When we rotate $\mathbb{R}^n \times \{t = 0\} \times \{u \geq 0\} \times \{v = 0\}$ around the axis $\mathbb{R}^n \times \{t = 0\} \times \{u = 0\} \times \{v = 0\}$, we rotate $\overline{P - D}$ as well. The result, denoted \tilde{P} , is called the *spun projection* of P .

Let \tilde{K} be an $(n + 1)$ -knot. Let \tilde{K} be the spun knot of an n -knot K . See [26] for a basic description of spun knots. Then we can suppose the following (1)(2).

(1) K is in $\mathbb{R}^n \times \mathbb{R}_t \times \{u \geq 0\} \times \{v = 0\}$.

(2) $K \cap (\mathbb{R}^n \times \mathbb{R}_t \times \{u = 0\} \times \{v = 0\})$ is the n -disc D which is defined above.

When we rotate $\mathbb{R}^n \times \mathbb{R}_t \times \{u \geq 0\} \times \{v = 0\}$ around the axis $\mathbb{R}^n \times \mathbb{R}_t \times \{u = 0\} \times \{v = 0\}$, we rotate $\overline{K - D}$ as well. The result is denoted \tilde{K} .

Lemma 3.1. *The above \tilde{P} is the projection of \tilde{K} .*

Proof. Let $\alpha_\theta : \mathbb{R}^n \times \mathbb{R}_t \times \mathbb{R}_u \times \mathbb{R}_v \rightarrow \mathbb{R}^n \times \mathbb{R}_t \times \mathbb{R}_u \times \mathbb{R}_v$ be the map $(*, *, u, v) \mapsto (*, *, u \cdot \cos\theta - v \cdot \sin\theta, u \cdot \sin\theta + v \cdot \cos\theta)$.

We can regard π as $\tilde{\pi}|(\mathbb{R}^n \times \mathbb{R}_t \times \{u = 0\} \times \{v = 0\})$.

Then $\tilde{\pi}|(\mathbb{R}^n \times \mathbb{R}_t \times \{u = r \cdot \cos\theta, v = r \cdot \sin\theta | r \geq 0, \theta \text{ is fixed}\})$ is equal to the compositions $(\alpha_\theta) \circ \tilde{\pi} \circ (\alpha_\theta^{-1})$.

Furthermore $\tilde{K} \cap (\mathbb{R}^n \times \mathbb{R}_t \times \{u = r \cdot \cos\theta, v = r \cdot \sin\theta | r \geq 0, \theta \text{ is fixed}\}) = \alpha_\theta(\overline{K - D})$

and $\tilde{P} \cap (\mathbb{R}^n \times \mathbb{R}_t \times \{u = r \cdot \cos\theta, v = r \cdot \sin\theta | r \geq 0, \theta \text{ is fixed}\}) = \alpha_\theta(\overline{P - D})$.

Hence $\tilde{\pi}(\tilde{K}) = \tilde{P}$. This completes the proof.

Let Q be the singular point set of P . Let \tilde{Q} be the singular point set of \tilde{P} . Then the following holds.

Lemma 3.2. *When we rotate $\mathbb{R}^n \times \{t = 0\} \times \{u \geq 0\} \times \{v = 0\}$ around the axis*

$\mathbb{R}^n \times \{t = 0\} \times \{u = 0\} \times \{v = 0\}$ to make \tilde{P} from P , we rotate Q as well. Then the result \tilde{Q} coincides with $Q \times S^1$ and \tilde{Q} consists of double points.

We use the above preliminaries to prove Theorem 1.3 when $n > 3$.

Let $P^{(3)}$ be the projection \hat{P} of a 3-knot, which is defined in the proof of the $r = 0$ case of Proposition 2.2. Let $P^{(n+1)}$ be the spun projection of $P^{(n)}$ ($n \geq 3$).

Claim 3.3. $P^{(n)}$ satisfies conditions (1)(2)(3) of Theorem 1.3. ($n \geq 3$).

Proof. By Proposition 2.2, there is a 3-knot $K^{(3)}$ which is diffeomorphic to the standard sphere and whose projection is $P^{(3)}$. Suppose $K^{(n+1)}$ is the spun knot of $K^{(n)}$ ($n \geq 3$). By induction on n , $K^{(n)}$ is diffeomorphic to the standard sphere ($n \geq 3$). By Lemma 3.1, $P^{(n)}$ is the projection of $K^{(n)}$. Hence $P^{(n)}$ satisfies condition (3) of Theorem 1.3.

Let $Q^{(n)}$ be the singular point set of $P^{(n)}$. By Lemma 3.2, $Q^{(n)}$ consists of double points. Hence $P^{(n)}$ satisfies condition (2) of Theorem 1.3.

In order to prove $P^{(n)}$ satisfies condition (1) of Theorem 1.3, we prove:

Lemma 3.4. *Let X be an $(n+1)$ -knot whose projection is $P^{(n+1)}$. Then there is an n -knot Y such that the projection of Y is $P^{(n)}$ and that X is the spun knot of Y . ($n \geq 3$.)*

Proof. By Lemma 3.2, The number of the connected components of $Q^{(n)}$ is μ . Let $Q_1^{(n)}, \dots, Q_\mu^{(n)}$ be the connected components of $Q^{(n)}$. Suppose $Q_i^{(n+1)}$ is made from $Q_i^{(n)}$ ($n \geq 3$).

Let A be an n -knot whose projection is P . Let $Q_{1i}^{(n)}, Q_{2i}^{(n)}$ be the connected components of $(\pi|A)^{-1}(Q_i^{(n)})$. Note that $Q_{1i}^{(n)} \subset A$ and $Q_{2i}^{(n)} \subset A$. We assign to the n -knot A an element $\rho(A) \in \mathbb{Z}_2^\mu$ as follows. Let $\mathbb{Z}_2 = \{+1, -1\}$. If $Q_{1i}^{(n)}$ is over (resp. under) $Q_{2i}^{(n)}$, then we define the i -th coordinate of $\rho(A)$ to be $+1$ (resp. -1). We define ‘over’ and ‘under’ by using the direction of \mathbb{R}_t . Note the following
(1) Let A' be another n -knot whose projection is P . If $\rho(A') = \rho(A)$, then A' is equivalent to A .
(2) For any element $x \in \mathbb{Z}_2^\mu$, there is such an n -knot K with $\rho(K) = x$.

Let X be an $(n+1)$ -knot whose projection is $P^{(n+1)}$. Then there is an n -knot Y whose projection is $P^{(n)}$ such that $\rho(Y) = \rho(X)$. Let \tilde{Y} be an $(n+1)$ -knot which is the spun knot of the n -knot X . By Lemma 3.1, the projection of \tilde{Y} is $P^{(n+1)}$. Then $\rho(\tilde{Y}) = \rho(Y)$. Hence $\rho(\tilde{Y}) = \rho(X)$, so the $(n+1)$ -knot X is equivalent to the $(n+1)$ -knot \tilde{Y} . By induction on n , Lemma 3.4 holds.

We continue the proof of Claim 3.3. If Lemma 3.5 below is true, then $P^{(n)}$ satisfies condition (1) of Theorem 1.3.

Lemma 3.5. *Let K be an n -knot whose projection is $P^{(n)}$. Then K is not unknotted. ($n \geq 3$.)*

Proof. The $n = 3$ case holds by §2. We prove $n > 3$. By Lemma 3.4, there are a 3-knot $Z^{(3)}$, a 4-knot $Z^{(4)}, \dots$, and an n -knot $Z^{(n)}$ such that $Z^{(q+1)}$ is the

spun knot of $Z^{(q)}$ ($q = 3, \dots, n-1$), that $Z^{(n)} = K$, and that the projection of $Z^{(r)}$ is $P^{(r)}$ ($r = 3, \dots, n$).

Recall that the following facts hold by Theorem 4.1 of [11] or by using the Mayer-Vietoris exact sequence. See, e.g., §14 of [19] for the Alexander polynomials. See, e.g., p.160 of [21] and [18] for the Alexander invariant. Let \tilde{X}_K denote the canonical infinite cyclic covering of the complement of the knot K .

Theorem 3.6 (known). *Let K be a simple $(2k+1)$ -knot ($k \geq 1$). Let $\Delta_K(t)$ be the Alexander polynomial of K . Suppose the $(k+1)$ -Alexander invariant $H_{k+1}(\tilde{X}_K; \mathbb{Q}) \cong \{\mathbb{Q}[t, t^{-1}] / \delta_K^1(t)\} \oplus \dots \oplus \{\mathbb{Q}[t, t^{-1}] / \delta_K^p(t)\}$. Then $\Delta_K(t) = a \cdot t^b \cdot \delta_K^1(t) \cdots \delta_K^p(t)$ for a rational number a and an integer b and we can put $\Delta_K(1) = 1$.*

Theorem 3.7 (known). *Let $K^{(n+1)}$ be the spun knot of $K^{(n)}$ ($n \geq 1$). Let $H_k(\tilde{X}_{K^{(n)}}; \mathbb{Q})$ (resp. $H_k(\tilde{X}_{K^{(n+1)}}; \mathbb{Q})$) denote the k -Alexander invariant of $K^{(n)}$ (resp. $K^{(n+1)}$). Suppose that $K^{(n)}$ bounds a Seifert hypersurface V such that $H_1(V; \mathbb{Z}) \cong 0$. Then $H_2(\widetilde{X^{(n+1)}}; \mathbb{Q}) \cong H_2(\widetilde{X^{(n)}}; \mathbb{Q})$.*

Proposition 3.8 (known). *Let $K^{(n+1)}$ be the spun knot of $K^{(n)}$ ($n \geq 1$). If $K^{(n)}$ has a simply connected Seifert hypersurface, then $K^{(n+1)}$ has a simply connected Seifert hypersurface.*

As in the proof of the $r=0$ case of Proposition 2.2, the Alexander polynomial $\Delta_{Z^{(3)}}(t)$ of $Z^{(3)}$ is nontrivial. By Theorem 3.6, the 2-Alexander invariant $H_2(\tilde{X}_{Z^{(3)}}; \mathbb{Q})$ is nontrivial. By Theorem 3.7 and Proposition 3.8, the 2-Alexander invariant $H_2(\tilde{X}_{Z^{(s)}}; \mathbb{Q})$ is nontrivial ($s = 4, \dots, n$).

Therefore this also completes the proof of Lemma 3.5. This completes the proof Theorem 1.3 in the final case when $n > 3$.

§4. THE PROOF OF THEOREM 4.1

An immersed n -sphere A is an image of a self-transverse immersion $S^n \looparrowright \mathbb{R}^{n+1}$. Let $\pi : \mathbb{R}^{n+1} \times \mathbb{R} \rightarrow \mathbb{R}^{n+1} \times \{0\}$ be the natural projection map. We regard an immersed n -sphere A as in $\mathbb{R}^{n+1} \times \{0\}$. We say that A lifts (into $\mathbb{R}^{n+1} \times \mathbb{R}$) if there is an n -knot K whose projection is A . Then K is called a lift.

It is natural to ask the following question.

Problem. Do all immersed n -spheres ($\subset \mathbb{R}^{n+1} \times \{0\}$) lift into $\mathbb{R}^{n+1} \times \mathbb{R}$?

It is well-known that, when $n = 1$, the answer is affirmative. [10] proved: when $n = 2$, the answer is negative. ([6] gave an alternative proof.)

In this section, we give the negative answer to the above Problem in the case $n > 2$.

Theorem 4.1. *Let n be any integer greater than two. There exists an immersed n -sphere ($\subset \mathbb{R}^{n+1} \times \{0\}$) which does not lift into $\mathbb{R}^{n+1} \times \mathbb{R}$.*

Note. Theorem 4.1 is different from Theorem 1.3.

Proof of Theorem 4.1. We define the *spun immersed* $(n+1)$ -sphere of an immersed n -sphere. Let A be an immersed n -sphere. Take $\mathbb{R}^n \times \{x|x \in \mathbb{R}\} \times \{y|y \in \mathbb{R}\}$. We suppose $A \subset \mathbb{R}^n \times \{x \geq 0\} \times \{y=0\}$. We suppose $A \cap (\mathbb{R}^n \times \{x=0\} \times \{y=0\})$ is an n -disc D embedded in $\mathbb{R}^n \times \{x=0\} \times \{y=0\}$. We suppose D does not intersect the singular point set of A . We regard $\mathbb{R}^n \times \{x|x \in \mathbb{R}\} \times \{y|y \in \mathbb{R}\}$ as the result of rotating

$\mathbb{R}^n \times \{x \geq 0\} \times \{y=0\}$ around the axis $\mathbb{R}^n \times \mathbb{R}_t \times \{x=0\} \times \{y=0\}$. When we rotate

$\mathbb{R}^n \times \{x \geq 0\} \times \{y=0\}$ around the axis $\mathbb{R}^n \times \mathbb{R}_t \times \{x=0\} \times \{y=0\}$, we rotate $\overline{A-D}$ as well. The result, denoted \tilde{A} , is called the *spun immersed* $(n+1)$ -sphere of A .

We regard $\mathbb{R}^n \times \{x|x \in \mathbb{R}\} \times \{y|y \in \mathbb{R}\}$ as

$$\mathbb{R}^n \times \{x|x \in \mathbb{R}\} \times \{y|y \in \mathbb{R}\} \times \{z|z=0\} \subset \mathbb{R}^n \times \{x|x \in \mathbb{R}\} \times \{y|y \in \mathbb{R}\} \times \{z|z \in \mathbb{R}\}.$$

We consider whether $\tilde{A} (\subset \mathbb{R}^n \times \{x|x \in \mathbb{R}\} \times \{y|y \in \mathbb{R}\} \times \{z=0\})$ lifts into $\mathbb{R}^n \times \{x|x \in \mathbb{R}\} \times \{y|y \in \mathbb{R}\} \times \{z|z \in \mathbb{R}\}$.

We consider whether $A (\subset \mathbb{R}^n \times \{x \geq 0\} \times \{y=0\} \times \{z=0\})$ lifts into $\mathbb{R}^n \times \{x \geq 0\} \times \{y=0\} \times \{z|z \in \mathbb{R}\}$.

We prove Claim 1. By the above result of [10] and Claim 1, Theorem 4.1 holds.

Claim 1. Let \tilde{A} be the spun immersed $(n+1)$ -sphere of an immersed n -sphere A . If \tilde{A} lifts, then A lifts.

Proof of Claim 1. Let E be an immersed m -sphere in $\mathbb{R}^{m+1} \times \{0\}$. Suppose E lifts into $\mathbb{R}^{m+2} = \mathbb{R}^{m+1} \times \mathbb{R}$. Let K be a lift of E . Let Q be a compact n -submanifold $\subset E$. Suppose Q does not intersect with the singular point set of E . By using the partition of unity, we can suppose that $E \cap K = Q$.

Let $\tilde{K} \subset \mathbb{R}^n \times \{x|x \in \mathbb{R}\} \times \{y|y \in \mathbb{R}\} \times \{z|z \in \mathbb{R}\}$ be a lift of $\tilde{A} \subset \mathbb{R}^n \times \{x|x \in \mathbb{R}\} \times \{y|y \in \mathbb{R}\} \times \{z=0\}$. We suppose $\tilde{A} \cap \tilde{K} = \partial D$.

Let $K_0 = \tilde{K} \cap (\mathbb{R}^n \times \{x \geq 0\} \times \{y=0\} \times \{z|z \in \mathbb{R}\})$. Then $K_0 \cap D$ is $\partial D = \partial K_0$.

Then $K = K_0 \cup D$ is an n -submanifold $\subset \mathbb{R}^n \times \{x \geq 0\} \times \{y=0\} \times \{z|z \in \mathbb{R}\}$. Then the projection of K is A . This completes the proof of Claim 1 and Theorem 4.1.

Theorem 4.2. Let $n > 2$. Let M be a closed manifold which can be immersed into $\mathbb{R}^{n+1} \times \{0\}$ and which can be embedded in $\mathbb{R}^{n+1} \times \mathbb{R}$. Then there exists an immersion $f : M \looparrowright \mathbb{R}^{n+1} \times \{0\}$ such that $f(M)$ does not lift into $\mathbb{R}^{n+1} \times \mathbb{R}$.

Proof. Take an immersion $g : M \looparrowright \mathbb{R}^{n+1} \times \{0\}$. Let A be an immersed n -sphere which does not lift into $\mathbb{R}^{n+1} \times \mathbb{R}$. Take a connected sum $g(M) \# A$ in $\mathbb{R}^{n+1} \times \{0\}$. If $g(M) \# A$ lifts, then A lifts. Hence $g(M) \# A$ does not lift. Take f so that $f(M) = g(M) \# A$.

APPENDIX. THE PROOF OF THEOREM 2.4.

We review the definition of the Alexander polynomial $\Delta_K(t)$ for a $(2k+1)$ -knot K . (See §14 of [19].) Let A be a Seifert matrix for K . We define $\Delta_K(t)$ to

be $\det(A - (-1)^k A')$, where A' is the transposed matrix. Note that we identify $\Delta_K(t)$ with $(-1)^r t^s \Delta_K(t)$ (for any $r, s \in \mathbb{Z}$).

Theorem 2.4 (known). *Let K be a simple $(2k+1)$ -knot ($k \geq 1$). Then the following conditions are equivalent.*

- (1) *The Alexander polynomial of the simple $(2k+1)$ -knot K is trivial.*
- (2) *The simple $(2k+1)$ -knot K is trivial.*

Note. See the definition of simple knots for §6 of [19].

Proof. (2) \Rightarrow (1) is obvious. We prove (1) \Rightarrow (2).

Let \tilde{X} be the infinite cyclic covering of the complement X of the knot K . By (1) we have: $\pi_1(X) \cong \mathbb{Z}$, $\pi_1(\tilde{X}) \cong 1$, and $H_i(\tilde{X}) \cong 0$ ($i > 0$).

By Hurewitz's theorem, we have: $\pi_i(\tilde{X}) \cong 0$ ($i > 1$).

Therefore $\pi_i(X) \cong 0$ ($i > 1$).

The following theorem is proved essentially in [17], [22], [24]. By this theorem, K is the trivial knot. This completes the proof.

Theorem. ([17][22][24]) *Let K_1 be an n -knot ($n \geq 3$) in S^{n+2} . Then K is trivial if and only if the following conditions hold.*

- (1) $\pi_1(S^{n+2} - K_1) \cong \mathbb{Z}$.
- (2) $\pi_i(S^{n+2} - K_1) \cong 0$ ($i > 1$).

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Acknowledgements. The author would like to thank Prof. A. Kawauchi, Prof. A. Ranicki, and Prof. J. Levine for explaining to him the full story behind Theorem 2.4. The author also would like to thank Prof. K. Igusa for correcting his English. Finally the author would like to thank the referee(s) and the editor(s) for reading the manuscript.

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